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Gauge theories on noncommutative spaces

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ABSTRACT. I review my results about noncommutative gauge theories and about the relation of these theories to M(atrix) theory following my lecture on ICMP 2000.

In my lecture on ICMP 2000 I gave a short review of my results on noncommutative gauge theories and talked in more detail about my recent paper [9]. Here I'll skip all details referring to papers [1]-[13]. I'll list only main results of these papers.

In the paper [1] it was shown that gauge theories on noncommutative tori appear naturally in consideration of compactifications of M(atrix) theory. The same logic can be used to obtain gauge theories on noncommutative toroidal orbifolds [14], [15], [11], [12].

More precisely, if G is a subgroup of the group of symmetries of any model we can restrict ourselves to fields that are G-invariant up to gauge equivalence. This means that the change of a field A under the action of an element $\gamma \in G$ can be compensated for by gauge transformation U_γ. For matrix models (i.e. in the case when A is a collection of matrices and gauge transformations are unitary transformations) this means that

$$\gamma(A) = U_\gamma A U_\gamma^{-1}. \tag{1}$$

Usually finite size matrices don't satisfy this equation; one should replace (Hermitian) matrices by (Hermitian) operators in infinite-dimensional Hilbert space E and consider U_γ as unitary operators in this space. There exists no reason to expect that $U_{\gamma\lambda} = U_\gamma U_\lambda$, but taking into account that

$$(U_{\gamma\lambda}^{-1} \cdot U_\gamma U_\lambda) A (U_{\gamma\lambda}^{-1} \cdot U_\gamma U_\lambda) = A \tag{2}$$

it is naturally to assume that

$$U_{\gamma\lambda} = e^{i\pi\theta(\gamma,\lambda)} U_\gamma U_\lambda \tag{3}$$

One can say, that the operators U_γ specify a projective representation of the group G. In the case when $G = \mathbf{Z}^d$ the associative algebra T_θ^d generated by operators U_γ can be interpreted as the algebra of functions on d-dimensional noncommutative torus. In other words the space E can be considered as a T_θ^d-module.

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We always consider finitely generated projective modules (direct summands in free modules $(T^d_\theta)^n$). In noncommutative geometry this means that we consider "vector bundles" over noncommutative tori.

The torus T_θ is specified by means of bilinear form $\theta(\gamma,\lambda)$ on $\mathbf{Z}^d$; without loss of generality one can assume that this form is antisymmetric. It will be more convenient for us to say that a noncommutative torus is determined by antisymmetric matrix θ_{jk} corresponding to the form $\theta(\gamma,\lambda)$ in some basis of $\mathbf{Z}^d$. In terms of this matrix noncommutative torus can be interpreted as an algebra with unitary generators $U_1,...,U_d$ satisfying relations $U_jU_k = e^{2\pi i\theta_{jk}}U_kU_j$. If $A = (A_1,...,A_d)$ and the group $\mathbf{Z}^d$ acts on A by means of translations (i.e. $\gamma(A) = A+\gamma$), then the solution of the equation (1) can be considered as a connection on noncommutative torus T_θ in the sense of A.Connes [16], [17]. (The notion of connection is discussed in detail at the end of the paper.) If our starting point is BFSS or IKKT matrix model [18], [19],then the above construction leads to SUSY Yang-Mills theory on noncommutative torus [1]. Replacing $\mathbf{Z}^d$ with semidirect product of $\mathbf{Z}^d$ and finite group we obtain gauge theories on noncommutative toroidal orbifolds. The appearance of noncommutative geometry can be explained not only from the viewpoint of M(atrix) theory, but also from the viewpoint of string theory as was shown in a series of papers [20]-[24], culminating by Seiberg-Witten paper [25] that contains very detailed analysis of relation between string theory and gauge theory on noncommutative spaces.

Gauge theories on noncommutative tori were studied by A.Connes and M. Rieffel, especially in two-dimensional case [26]-[28]. I obtained new results about these theories focusing my attention on problems related to physics. Already in [1] it was conjectured that Morita equivalence of algebras is related to duality in physics. One says that an algebra A is Morita equivalent to the algebra $\hat{A}$ if the category of A-modules is equivalent to the category of $\hat{A}$-modules. In other words, we should be able to transfer A-modules into $\hat{A}$-modules and $\hat{A}$-modules into A-modules; this correspondence should be natural (for every A-linear map $\varphi : E \to E'$ of A-modules should be defined an $\hat{A}$-linear map $\hat{\varphi} : \hat{E} \to \hat{E}'$ of corresponding $\hat{A}$-modules; one requires that the correspondence $\varphi \to \hat{\varphi}$ transforms composition of maps into composition of maps). However, to prove that gauge theories over A are related to gauge theories on $\hat{A}$ we should be able to transfer also connections on A-modules to connections on $\hat{A}$-modules. I introduced a new notion of gauge Morita equivalence (in original paper [2] I used the term "complete Morita equivalence") and proved that gauge Morita equivalence of algebras implies physical equivalence of corresponding gauge theories.It is proved in [2] that noncommutative tori T^d_θ and $T^d_{\hat{\theta}}$ are gauge Morita equivalent if and only if there exists a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{4}$$

belonging to $SO(d,d,\mathbf{Z})$ and obeying

$$\hat{\theta} = (A\theta + B)(C\theta + D)^{-1} \tag{5}$$

Here A,B,C,D are $d\times d$ matrices and $SO(d,d,\mathbf{Z})$ stands for the group of $2d\times 2d$ matrices with integer entries that are orthogonal with respect to quadratie form $x_1x_{d+1} + ... + x_dx_{2d}$ having signature (d,d). (The fact that the relation (5) implies Morita equivalence was proved in earlier paper [4] written together with M. Rieffel.)

Equivalence of gauge theories on noncommutative tori T_θ and $T_{\hat{\theta}}$ was studied in detail in [2], [6], [7], [8]. It is closely related to T-duality in string theory; this relation was thoroughly analyzed in [5]. This analysis led, in particular, to the discovery of possibility to trade noncommutativity parameter for background field in the expressions for BPS energies. (Almost simultaneously this fact was found in [25] at the level of action functionals; it was called background independence.)

The papers [6], [7], [8], [13] are devoted to the study of BPS fields and BPS states in SUSY gauge theories on noncommutative tori. Analysis of BPS spectra by means of supersymmetry algebra was performed in [7]. Another way to study BPS states is based on geometric quantization of moduli spaces of classical configurations having some supersymmetry (BPS fields). One can identify $\frac{1}{2}$ BPS fields with connections having constant curvature. We found necessary and sufficient for existence of constant curvature connections and described moduli spaces of such connentions. As in the case of commutative torus the K-theory class of a projective module E (of a "vector bundle") over T^d_θ can be characterized by means of an even integer element $\mu(E)$ of Grassman algebra with d generators $\alpha^1, ..., \alpha^d$ (as a collection of integer antisymmetric tensors $\mu^{(0)}, \mu^{(2)}_{ij}, \mu^{(4)}_{ijkl}, ...$ of even rank). One can interpret $\mu^{(0)}, \mu^{(2)}_{ij}, ...$ as numbers of D-branes with given topological charges (although a notion of individual D-brane is ill-defined in noncommutative space). We will assume that θ is irrational; then it was proved in [27] that two modules having the same K-theory class are isomorphic. One can verify that necessary and sufficient condition for existence of constant curvature connection in a module with $\mu(E) = \mu$ is a possibility to represent μ as a quadratic exponent (as an expression of the form $Cexp(\alpha^k \rho_{kl} \alpha^l)$) or as a limit of quadratic exponents. If a module E having constant curvature connection cannot be represented as a direct sum of isomorphic modules (i.e. g.c.d.$(\mu^{(0)}, \mu^{(2)}_{ij}, \mu^{(4)}_{ijkl}, ...) = 1$) we say that E is a basic module. For basic T^d_θ-module the moduli space of constant curvature connections is a d-dimensional torus T^d. An arbitrary T^d_θ-module with constant curvature connection is isomorphic to a direct sum of n basic modules where $n =$g.c.d.$(\mu^{(0)}, \mu^{(2)}_{ij}, \mu^{(4)}_{ijkl}, ...)$; for such a module corresponding moduli space is $(T^d_\theta)^n/S_n$. Basic modules and constant curvature connections on these modules can be described very explicitly in the language of so called Heisenberg modules. Every T^d_θ-module can be represented as a direct sum of basic modules, the number of summands in this sum can be made as large as we want. This statement follows from results of [27]. (Recall that we assumed irrationality of θ; this assumption is necessary and sufficient for validity of our claim.) This means, in particular, that every combination of D-branes can decay into $\frac{1}{2}$BPS states.

Instantons on noncommutative $\mathbf{R}^4$ were analyzed in [3] by means of generalization of ADHM construction. The most striking feature of noncommutative instantons is the absence of small instanton singularity in moduli space of noncommutative instantons. Instantons on noncommutative tori were studied in [10]; in particular, we constructed a noncommutative analog of Nahm transform. Instantons can be characterized as $\frac{1}{4}$BPS fields. In the case when a T^d_θ-module admits constant curvature connection it is possible to give a complete description of $\frac{1}{4}$BPS fields and $\frac{1}{4}$BPS states [6]. To obtain this description one can apply the fact that under certain conditions on θ we can use Morita equivalence to transform such a module into a free module over another noncommutative torus. (It is sufficient to

assume that every linear combination of matrix elements of θ having integer coefficients is irrational.) This remark can be used also in many other cases; it confirms the idea that noncommutative tori with irrational θ are simpler that commutative tori.

Gauge theories on noncommutative toroidal orbifolds were studied in [11], [12]. Fairly complete analysis of modules, of constant curvature connections and corresponding moduli spaces, of Morita equivalence is given for $T^d_\theta/\mathbf{Z}_2$; however, the methods developed in [11], [12] work also for other toroidal orbifolds.

All results we mentioned are based on the notion of connection on A-module. There exist different definitions of this notion, but all of them are based on the same idea: a connection should satisfy Leibniz rule. If an n-dimensional Lie algebra L acts on associative algebra A by means of infinitesimal automorphisms (derivations) we can define a connection on (left) A-module E as a collection of n linear operators $\nabla_i : E \to E, \quad i = 1, ..., n$ obeying the Leibniz rule:

$$\nabla_i(ae) = a \cdot \nabla_i e + \delta_i a \cdot e,$$

where $a \in A, \quad e \in E$, and $\delta_1, ..., \delta_n$ are derivations corresponding to elements of a basis of Lie algebra L. (Notice, that operators ∇_i don't commute with multiplication by $a \in A$, i.e. they are $\mathbf{C}$-linear, but not A-linear. However, if ∇_i , ∇'_i $(i = 1, ..., n)$ are two connections the difference $\nabla'_i - \nabla_i$ is A-linear ; in other words $\nabla'_i - \nabla_i$ is an endomorphism of E.)

When we consider noncommutative tori we should define connections using d-dimensional commutative Lie algebra acting on T^d_θ by means of translations.

If we would like to define connections in terms of covariant differential instead of covariant derivative we should assume that the algebra A is a $\mathbf{Z}_2$-graded associative algebra equipped with a parity reversing derivation $Q : A \to A$. The standard assumption is that $Q^2 = 0$ (then A is called a graded differential algebra). However it is shown in [9] that one can relax this assumption requiring only that $Q^2 a = [\omega, a]$. (Here ω is a fixed element of A obeying $Q\omega = 0$.) If A is an associative algebra equipped with an operator Q of this kind (a Q-algebra is the terminology of [9]) we can define a connection on (left) A-module E as an linear operator $\nabla : E \to E$ obeying the Leibniz rule:

$$D(ae) = (-1)^{deg a} a De + Qa \cdot e.$$

The standard theory of connections (including the notion of Chern character) can be generalized to the case of modules over a Q-algebra. If P is a module over Q-algebra A and ∇_P is a connection on P we can define a structure of Q-algebra on $\hat{A} = \mathrm{End}_A P$ by the formula $\tilde{Q}\varphi = [\nabla_P, \varphi]$. (Here $\mathrm{End}_A P$ stands for an algebra of endomorphisms of A-module P, i.e. for an algebra of A-linear maps of P into itself.) Under certain conditions on P the algebra $\hat{A}$ is Morita equivalent to A, i.e. we can transfer A-modules into $\hat{A}$-modules and vice versa. (If A has a unit we should require that A considered as left A-module is a direct summand of P^N for some N and P is projective) Using the connection ∇_P we can transfer connections on A-modules to connections on corresponding on $\hat{A}$-modules. This operation permits us to extend the equivalence between categories of A-modules and $\hat{A}$-modules to an equivalence of corresponding gauge theories . This gives a very general duality theorem; $SO(d, d, \mathbf{Z})$ duality of gauge theories on noncommutative tori can be derived from this general theorem [9].

References.

1. A. Connes, M. Douglas, and A. Schwarz, *Noncommutative Geometry and Matrix Theory: Compactification on Tori*, JHEP **02** (1998), 3-38

2. A. Schwarz, *Morita Equivalence and Duality*, Nucl. Phys. **B 534** (1998), 720-738.

3. N. Nekrasov and A. Schwarz, *Instantons on Noncommutative R^4 and (2,0) Superconformal Six-dimensional Theory*, Comm. Math. Phys.**198** (1998), 689-703.

4. M. Rieffel and A. Schwarz, *Morita Equivalence of Multidimensional Noncommutative Tori*, Intl. J. of Math **10(2)** (1999), 289-299

5. B. Pioline and A. Schwarz, *Morita Equivalence and T-Duality*, JHEP **9(21)** (1999), 1-16

6. A. Konechny and A. Schwarz, *1/4 - BPS States on Noncommutative Tori*, JHEP. **30** (1999), 1-14

7. A. Konechny and A. Schwarz, *Supersymmetry Algebra and BPS States of super Yang-Mills Theories on Noncommutative Tori*, Phys. Lett. **B 453** (1999), 23-29

8. A. Konechny and A. Schwarz, *BPS States on Noncommutative Tori and Duality*, Nucl. Phys. **B 550** (1999), 561-584.

9. A. Schwarz, *Noncommutative Supergeometry and Duality*, Lett. Math. Phys. **50 (4)** (1999), 309-321

10. A. Astashkevich, N. Nekrasov and A. Schwarz, *On Noncommutative Nahm Transform*, Comm. Math. Phys. **211 (1)** (2000), 167-182.

11. A. Konechny and A. Schwarz, *Moduli Spaces of Maximally Supersymmetric Solutions on Noncommutative Tori and Noncommutative Orbifolds*, JHEP **09** (2000), 1-23.

12. A. Konechny and A. Schwarz, *Compactification of M(atrix) Theory on Noncommutative Toroidal Orbifolds*, Nucl. Phys. **B 591(3)** (2000), 667-684.

13. A. Astashkevich and A. Schwarz, *Projective Modules Over Noncommutative Tori: Classification of Modules with Constant Curvature Connection*, Journal of Operator Theory (in press).

14. M. Douglas, *D-branes and Discrete Torsion*, hep-th/9807235.

15. P.M. Ho, Y.Y. Wu and Y.S. Wu, *Towards Noncommutative Geometric Approach to Matrix Compactification*, Phys.Rev. **D 58** (1998), 26006.

16. A. Connes, *C*-algebres et Geometrie Differentielle*, C. R. Acad. Sci. Paris **290** (1980), 599-604.

17. A. Connes, *Noncommutative Geometry*, Academic Press, 661 pp.

18. T. Banks, W. Fischler, Shenker, and I.Susskind, *M-theory as a Matrix Model: a Conjecture*, Phys. Rev. **D 55** (1997), 5112.

19. N. Ishibashi, H. Kawai, I. Kitazawa, and Tsuchiya, *A Large-N educed Model as Superstring*, Nucl. Phys. **B 492** (1997), 467-491.

20. M. Douglas and C. Hull, *D-branes and Noncommutative Torus*, JHEP **02** (1998), 008

21. Y.-K. E. Cheung and M. Krogh, *Noncommutative Geometry from 0-Branes in a Background B Field*, Nucl. Phys. **B 528** (1998), 185.

22. C.-S. Chu and P.-M. Ho, *Noncommutative Open String and D-Brane*, Nucl. Phys. **B550** (1999), 151; *Constrained Quantization of Open String in Background B Field and Noncommutative D-Brane*, Nucl. Phys. **B 568** (2000), 447.

23. V. Schomerus, *D-Branes and Deformation Quantization* , JHEP **06** (1999), 030.

24. F. Ardalan, H. Arfaei and M. Sheikh-Jabbari, *Mixed Branes and M(atrix) Theory on Noncommutative Torus*, PASCOS 98; *Noncommutative Geometry from String and Branes*, JHEP **02** (1999), 016; *Dirac Quantization of Open Strings and Noncommutativity in Branes*, Nucl.Phys. **B 576** (2000), 578-596.

25. N. Seiberg and E. Witten, *String Theory and Noncommutative Geometry*, JHEP **09** (1999), 032.

26. A. Connes and M. Rieffel, *Yang-Mills for Noncommutative Two-Tori*, Contemporary Math. **66** (1987), 237-266.

27. M. Rieffel, *Projective Modules over Higher-dimensional Noncommutative Tori*, Can. J. Math., Vol. **XL**, No. 2 (1998), 257-338.

28. M. Rieffel, *Critical Points of Yang-Mills for Noncommutative Two-Tori*, J. Diff. Geom., **31** (1990), 535.

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